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THE RIGIDITY OF THE SMALLEST UNKNOWN π-SPACE

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A π-space is a planar space all of whose planes are isomorphic to a given linear space π. Examples are given by projective and affine spaces. The smallest linear space π having lines of different sizes for which the existence of a π-space is still unsolved has 7 points and the corresponding π-space would be a regular space of 47 points. We prove that such a π-space is rigid, i.e. has no other automorphism than the identity.

1. Introduction

A *linear space* is a set of elements called *points* together with a family of subsets called *lines*, such that every pair of points is in exactly one line, each line containing at least two points. A *subspace* of a linear space L is a subset L' such that any line of L having at least two points in L' is contained in L'. A *planar space* is a linear space provided with a family of distinguished subspaces called *planes*, such that every triple of non-collinear points is in exactly one plane, each plane containing at least three non-collinear points.

A *π-space* is a planar space all of whose planes are isomorphic to a given linear space π. This notion was introduced by Buekenhout and Deherder [2] and studied by Brouwer [1] and Leonard [4]. π-spaces are always assumed to contain at least two planes (otherwise they are called *trivial*). A π-space having a finite number of points is said to be *finite*.

The planar spaces consisting of $2k$ points lying on two disjoint lines of k points, all other lines having two points, are at present the only known examples of non-trivial finite π-spaces having lines of different sizes (note that their planes are degenerate projective planes). In [3], some rather restrictive relations on the parameters of such spaces are given. The smallest linear space for which the existence of a non-trivial finite π-space with different line sizes is unsettled is the linear space π_1 given by Fig. 1 (the lines of size 2 are not represented).

We shall say that a point x of π_1 is of *degree* 1, 2 *or* 3 according as it belongs to exactly 1, 2 or 3 lines of 3 points. The unique point of degree 3 in π_1 will be called the *top* of π_1. Two points x and y will be called *adjacent* (resp. *non-adjacent*) if the line $\langle x, y\rangle$ has size 3 (resp. 2).

We have shown in [3] that a non-trivial finite π_1-space S has necessarily 47 points and that every point is on exactly 12 (resp. 22) lines of size 3 (resp. 2) and

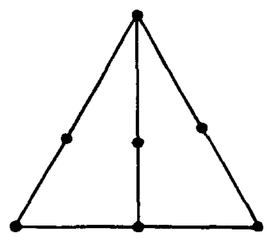

Fig. 1. π_1.

is the top of exactly 11 planes. Moreover the arrangement of the 9 planes containing a line of size 2 is independent from the choice of this line. More precisely, if x and y are two non adjacent points of S, then the set of all points which are not adjacent to x or y is partitioned into 3 pairs of points collinear with x, generating with y a plane in which y is of degree 2 and x of degree 1; 3 pairs of points collinear with y, generating with x a plane in which x is of degree 2 and y of degree 1; 3 points which are each the top of the plane that they generate with x and y.

Theorem. *If S is a non-trivial finite π_1-space, then S is rigid, i.e. has no other automorphism than the identity.*

Hence the construction of a non-trivial π_1-space (if there exists one) would probably not be so easy.

In what follows, a line or a plane which is invariant by a permutation α is said to be fixed by α iff each of its points is fixed by α.

2. Proof

If S has an automorphism α distinct from the identity, α generates a cyclic group of order $O(\alpha)$ divisible by a prime number p, and so by the theorem of Sylow–Cauchy, there is an element of order p in Aut(S). Therefore it suffices to prove that, for every prime number p, S has no automorphism of order p.

Suppose on the contrary that there is such an automorphism α. The set F of its fixed points is a linear subspace of S. Moreover, if F contains three non-collinear points, the plane π generated by these three points is invariant under α and α induces in this plane an automorphism fixing at least three non-collinear points. But the only automorphism of π_1 fixing three non-collinear points is the identity. This proves that F is a π_1-space contained in S. Therefore, since a non-trvial π_1-space has exactly 47 points, F is either a plane or S itself.

We conclude that the set F of fixed points of α is the empty set, a single point, a line or a plane.

– If F is empty, then α is a cycle of length 47.

– If F consists of a single point, the 46 remaining points of S are on 2 cycles of length 23, or on 23 cycles of length 2 of α.

– If F is a line of two points, the 45 remaining points are on 9 cycles of length 5, or on 15 cycles of length 3 of α

– If F is a line of three points, the 44 remaining points are on 4 cycles of length 11, or on 22 cycles of length 2 of α

– If F is a plane, the 40 remaining points are on 8 cycles of length 5, or on 20 cycles of length 2 of α.

We will successively rule out each of these possibilities.

1. *Suppose* α *is a cycle of length* 47

We may identify the points of S with the elements $\mathbb{Z}_{47}$, so that α corresponds to the permutation $x \to x+1 \pmod{47}$. Every line of S being the image of a line passing through 0 by some power of α, let us consider more closely the lines through 0.

The line $\{0, a, b\}$ determines three pairs of opposite elements of $\mathbb{Z}_{47}$, namely $\pm a$, $\pm(b-a)$, $\pm b$, which we shall call the *distances* between the points of this line. These three distances are non zero and pairwise distinct. Indeed if two distances were equal, the image of the line $\{0, a, b\}$ by some power of α would be of the form $\{0, x, 2x\}$ $(x \neq 0)$ and its image by α^x would be $\{x, 2x, 3x\}$. Since each of these two lines contains the points x and $2x$, they must coincide, and so $0 = 3x$ $(x \neq 0)$, which is impossible in $\mathbb{Z}_{47}$.

It follows that if $\{0, a, b\}$ is a line, its images by α^{-a} and α^{-b}, namely $\{-a, 0, b-a\}$ and $\{-b, a-b, 0\}$, are two other lines containing 0. But there must be 12 lines of 3 points through 0, that is 4 'well chosen' lines $\{0, a_i, b_i\}$ and their images by the various powers of α. Moreover all the distances defined by these 4 lines must be distinct. Therefore, to the set of lines of size 3, are associated 12 of the 23 possible distances between the points of S.

We will now show that if d is a distance between two adjacent points, then $2d$ is not a distance between two adjacent points. In order to do this, consider the plane generated by the lines $\{0, a, b\}$ and $\{-b, -b+a, 0\}$. It contains also the line $\{-b+a, -b+2a, a\}$ and has to be isomorphic to π_1. In particular, $\{-b, -b+2a\}$, $\{-b, b\}$ and $\{b, -b+2a\}$ must be lines, and so $\pm 2a$, $\pm 2b$ and $\pm 2(b-a)$ must be distances between non-adjacent points.

But in the multiplicative group of $\mathbb{Z}_{47}$, the multiplication by 2 is a fixed point free permutation of order 23. One of its cycles of length 23 contains the powers of 2, the other contains their opposites. Distances being defined up to sign, they are represented by the elements of one of these cycles. Thus the distances associated with the lines of size 3 are 12 of the 23 elements of this cycle and must, by the preceding statement, be arranged so that two of them are never consecutive, which is impossible.

Remark. A similar argument can be used to prove that no line of size 3 is contained in a cycle of length 11 (this result will be used later). Indeed, suppose that a cycle C of length 11 contains a line L of size 3. We may identify the points of C with the elements of $\mathbb{Z}_{11}$. Remember that if d is a distance between two points of L, then $2d$ is not such a distance. Therefore, among the 5 distances between the points of C, there are 3 distances such that none is twice another, which is impossible.

2. *Suppose that* α *has* 2 *cycles of length* 23

The fixed point x is on exactly 22 lines of two points, but the 23 distinct images of such a line by the various powers of α are all lines of size 2 containing x, a contradiction.

3. *Suppose that* α *has* 4 *cycles of length* 11

Let us denote the three collinear fixed points of α by a, b, c and the 4 cycles of length 11 by

$$A=\{a_i \mid i\in\mathbb{Z}_{11}\},\qquad B=\{b_i \mid i\in\mathbb{Z}_{11}\},$$
$$C=\{c_i \mid i\in\mathbb{Z}_{11}\},\qquad D=\{d_i \mid i\in\mathbb{Z}_{11}\}$$

where $\alpha(a_i)=a_{i+1}$, $\alpha(b_i)=b_{i+1}$, $\alpha(c_i)=c_{i+1}$, $\alpha(d_i)=d_{i+1}$ for every $i\in\mathbb{Z}_{11}$.

3.1. *We first show that the* 12 *lines of size* 3 *through* a (*resp.* b, c) *are* $\{a, b, c\}$ *and the* 11 *images of* $\{a, a_0, d_0\}$ (*resp.* $\{b, b_0, d_0\}$, $\{c, c_0, d_0\}$) *by the various powers of* α.

A line of 3 points not invariant by α and containing one of the fixed points of α, must have its points in distinct cycles of length 11, otherwise some pair of points would be in two distinct lines (because 11 is odd). Since there are only 4 cycles of length 11, there are necessarily two lines of 3 points intersecting in a point d_0 of one of the cycles of length 11; moreover, each of these two lines contains one of the three fixed points (we may assume without loss of generality that a and c are these fixed points). Thus $\langle a, d_0\rangle=\{a, d_0, x\}$ and $\langle c, d_0\rangle=\{c, d_0, y\}$. Suppose that the points x and y belong to the same cycle and consider the plane $\langle d_0, a, c\rangle$. The lines $\langle a, y\rangle$ and $\langle c, x\rangle$ are contained in this plane and are respectively the images of $\langle a, x\rangle$ and $\langle c, y\rangle$ by some powers of α. Thus, in this plane, each of the points a and c is on 3 lines of size 3, which is a contradiction. Therefore x and y belong to two distinct cycles, different from D. Let us now denote x and y by a_0 and c_0 respectively. The 12 lines of 3 points containing a (resp. c) are $\{a, b, c\}$ and the 11 images of $\{a, a_0, d_0\}$ (resp. $\{c, c_0, d_0\}$) by the various powers of α. Thus the lines $\langle a, c_0\rangle$ and $\langle c, a_0\rangle$ are of size 2. Since every plane is isomorphic to π_1, $\langle b, d_0\rangle$ is a line of size 3 and we may call its third point b_0. Indeed this point obviously does not belong to D and if it belongs to A or C, we would show as before that the plane $\langle a, b, d_0\rangle$ contains two points on 3 lines of size 3. This proves the announced result.

3.2. *There is a line containing one point of D and two points of one of the cycles A, B, C.*

The preceding argument shows that a fixed point cannot be the top of a plane containing the line $\{a, b, c\}$. Therefore, a is the top of 11 planes not containing $\{a, b, c\}$. These planes are the images of a plane containing the lines $\{a, x_0, y_0\}$, $\{a, x_i, y_i\}$, $\{a, x_j, y_j\}$, $\{x, x_i, y_j\}$, where either x_0, x_i, $x_j \in A$ and y_0, y_i, $y_j \in D$, or $x_0, x_i, x_j \in D$ and $y_0, y_i, y_j \in A$; indeed the remark in 1 shows that no line of size 3 is contained in a cycle of length 11, which implies that neither $\{x_0, x_i, x_j\}$ nor $\{y_0, y_i, y_j\}$ are lines. The three distances $\pm i, \pm j, \pm(i-j)$ between the points of D belonging to the same plane of top a are pairwise distinct. Indeed, $i \neq j$ (resp. $i \neq i-j$, $j \neq j-i$) is obvious and if $i=-j$ (resp. $i=j-i$, $j=i-j$), then $\alpha^{-i}\langle x_0, x_i\rangle = \langle x_j, x_0\rangle$ (resp. $\alpha^{i}\langle x_0, x_i\rangle = \langle x_i, x_j\rangle$, $\alpha^{j}\langle x_0, x_j\rangle = \langle x_j, y_i\rangle$) would be a line of size 3, which is not true. Moreover at least two of these three distances are distances between non adjacent points of D. By considering the planes of top b (resp. c), we get similar results.

This proves in particular that there are 3 lines of size 3, say L, L', L'', which are contained respectively in $D \cup A$, $D \cup B$, $D \cup C$. Let us prove that at least one of these lines contains only one point of D. Suppose, on the contrary, that each of the lines L, L', L'' contains two points of the cycle D and let δ_i $(i=1, \dots, 5)$ denote the 5 distances between the points of D. Let δ_1 (resp. δ_2, δ_3) be the distance between the two points of $L \cap D$ (resp. $L' \cap D, L'' \cap D$). Obviously, these three distances are distinct. Then the three distances between the points in a same plane of top a (resp. b and c) are $\delta_1, \delta_4, \delta_5$ (resp. $\delta_2, \delta_4, \delta_5$ and $\delta_3, \delta_4, \delta_5$). But there is a relation between the distances determined by three points, and so

$$\delta_1 = \delta_4 + \delta_5 \quad \text{or} \quad \delta_4 - \delta_5,$$
$$\delta_2 = \delta_4 + \delta_5 \quad \text{or} \quad \delta_4 - \delta_5,$$
$$\delta_3 = \delta_4 + \delta_5 \quad \text{or} \quad \delta_4 - \delta_5.$$

It follows that $\delta_1, \delta_2, \delta_3$ cannot be 3 distinct distances, a contradiction.

3.3. *There is no line of* 3 *points contained in the union of two of the cycles A, B, C.*

We already know that no line of 3 points is contained in a cycle of length 11. Suppose that there is a line of 3 points contained in the union of two of the cycles A, B, C. Without any loss of generality, we may assume that this line is $\{a_0, a_i, b_j\}$. The point c is non-adjacent to each of the three points of this line, which is impossible in a plane isomorphic to π_1.

Thanks to 3.2, we know that one of the lines L, L', L'' contains only one point of the cycle D. We may assume that so is the line L. This implies that the three distances $\delta_1, \delta_2, \delta_3$ between the points of D lying in a plane of top a are distances between non adjacent points of D.

Let $\pm j \neq \delta_1, \delta_2, \delta_3$. The top of the plane $\langle a, d_0, d_j\rangle$ cannot be a, and so it must be a_0, a_j, d_0 or d_j. Let us show that it is d_0 or d_j. The line $\langle a_0, a_j\rangle$ has no point in common with D. Indeed, if on the contrary $\langle a_0, a_j\rangle \cap D = \{d_i\}$, a would belong to the three lines of 3 points $\langle a, d_0\rangle$, $\langle a, d_j\rangle$, $\langle a, d_i\rangle$ contained in this plane and would therefore be the top of this plane. $\langle a_0, a_j\rangle$ has no point in common with A, B or C thanks to 3.3, and $\langle a_0, a_j\rangle$ has no fixed point thanks to 3.1. Thus $\langle a_0, a_j\rangle$ has only two points and the top of the plane $\langle a, d_0, d_j\rangle$ is either d_0 or d_j, which means that $\langle d_0, d_j\rangle$ is a line of size 3. The third point of $\langle d_0, d_j\rangle$ cannot be fixed and cannot belong to D or A (otherwise a would belong to three lines of 3 points and would therefore be the top of the plane $\langle a, d_0, d_j\rangle$). It follows that the third point of $\langle d_0, d_j\rangle$ belongs to B or to C. We may assume that $\langle d_0, d_j\rangle = \{d_0, d_j, b_k\}$ for a certain k.

The 11 planes whose top is b are the 11 images of the plane $\langle d_0, d_j, b\rangle$ by the various powers of α. Since $\pm j \neq \delta_1, \delta_2, \delta_3$, at least one of the three distances $\delta_1, \delta_2, \delta_3$ is distinct from the three distances $\pm j$, $\pm k$ and $\pm(k-j)$, let n denote this distance. So d_0 and d_n are non adjacent and the top of the plane $\langle b, d_0, d_n\rangle$ cannot be b or d_0 or d_n, therefore $\langle b_0, b_1\rangle$ is a line of size 3. By 3.1, the third point of this line cannot be fixed by α and, by 3.3, it cannot belong to A, B or C. Therefore it must belong to D; let us denote it by d_m. Then b is on the three lines of 3 points $\langle b, d_0\rangle$, $\langle b, d_n\rangle$, $\langle b, d_m\rangle$ and is the top of $\langle b, d_0, d_n\rangle$, a contradiction.

This proves that a π_1-space has no automorphism consisting of 4 cycles of length 11 and 3 fixed points.

4. *Suppose that α has* 9 *cycles of length* 5

We know that the set F of fixed points of α is a line of size 2. Let x be a point of F. The number of lines of size 3 not invariant by α and containing x is 12, which is not divisible by 5. This contradicts the fact that α is of order 5.

5. *Suppose that α has* 8 *cycles of length* 5

Then F is a plane. The number of lines of size 3 not invariant by α and containing the top of the plane F is $12-3=9$, which is not divisible by 5, a contradiction.

6. *Suppose that α has* 15 *cycles of length* 3

Then F is a line $\{a, b\}$ of size 2. Note first that the points d_0, d_1, d_2 of a cycle of length 3 are either collinear or pairwise non-adjacent, since otherwise α would leave the plane $\langle d_0, d_1, d_2\rangle$ invariant and induce in it an automorphism (distinct from the identity) permuting cyclically 3 points among which only two are adjacent, which is impossible.

The point a is on exactly 12 lines of size 3 and 22 lines of size 2. We conclude that a is joined to 8 cycles of length 3 by lines of size 3 and to 7 cycles of length 3 (denoted by C_i with $i = 1, \ldots, 7$) by lines of size 2. The three points of a cycle C_i cannot be collinear, otherwise the plane generated by a and this line would contain a point non-adjacent to each of the three points of a line, a situation which does not occur in π_1. Thus every cycle C_i contains 3 lines of size 2.

The planes generated by the (non-collinear) points of a cycle C_i are invariant by by α and the unique top of each of these planes is fixed. If a belongs to such a plane, a would be on 3 lines of size 2 in this plane, and so could not be the top of this plane. Therefore each of these planes contains b (the top) as well as the points of C_i and a line of size 3 invariant by α. Since the cycles C_i are pairwise disjoint, this implies that b is on at least $21 > 12$ lines of 3 points, a contradiction.

7. *Suppose that α is an involution*

We must rule out the cases:

- F is a point and α has 23 cycles of length 2.
- F is a line of size 3 and α has 22 cycles of length 2.
- F is a plane and α has 20 cycles of length 2.

Suppose that there is a fixed point belonging to two lines L_1 and L_2 of size 3 invariant (but not fixed) by α. Since the plane $\langle L_1, L_2 \rangle$ is isomorphic to π_1, it contains a line L of size 3 intersecting $L_1 \cup L_2$ in two distinct points and, since it is invariant by α, it contains the image $\alpha(L)$ of this line, which intersects also $L_1 \cup L_2$ in two distinct points, different from the two preceding ones. Whether or not the lines L and $\alpha(L)$ intersect, this plane contains at least 5 points belonging to at least two lines of size 3, which is impossible in π_1.

We conclude that a fixed point belongs to at most one line of 3 points invariant (but not fixed) by α. Since α has at least 20 cycles of length 2, since F has at most 7 points and since the lines containing two points of the same cycle are either lines of size 2 or lines of size 3 whose third point is fixed by α, there are at least 13 cycles which are lines of size 2.

The only automorphisms of order 2 of π_1 fix a line through the top and permute the other two lines through the top. Thus a plane of S invariant (but not fixed) by α contains only one cycle whose points form a line. On the other hand, such a cycle and a fixed point generate a plane invariant (but not fixed) by α. So every fixed point belongs to at least 13 distinct planes invariant by α, in which this fixed point is on a line consisting of 3 fixed points. Therefore the case where F is reduced to a single point is ruled out. It follows that there is a point lying on only one line of size 3 contained in F and that this line is contained in at least 13 distinct planes invariant by α. This contradicts the fact that every line of size 3 of S is contained in exactly $(47-3)/(7-3) = 11$ distinct planes, which ends the proof.

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